

# Plane symmetric traversable wormholes in an anti-de Sitter background

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We construct solutions of plane symmetric wormholes in the presence of a negative cosmological constant by matching an interior spacetime to the exterior anti-de Sitter vacuum solution. The spatial topology of this plane symmetric wormhole can be planar, cylindrical and toroidal. As usual the null energy condition is necessarily violated at the throat. At the junction surface, the surface stresses are determined. By expressing the tangential surface pressure as a function of several parameters, namely, that of the matching radius, the radial derivative of the redshift function and of the surface energy density, the sign of the tangential surface pressure is analyzed. We then study four specific equations of state at the junction: zero surface energy density, constant redshift function, domain wall equation of state, and traceless surface stress-energy tensor. The equation governing the behavior of the radial pressure, in terms of the surface stresses and the extrinsic curvatures, is also displayed. Finally, we construct a model of a plane symmetric traversable wormhole which minimizes the usage of the exotic matter at the throat, i.e., the null energy condition is made arbitrarily small at the wormhole throat, while the surface stresses on the junction surface satisfy the weak energy condition, and consequently the null energy condition. The construction of these wormholes does not alter the topology of the background spacetime (i.e., spacetime is not multiply-connected), so that these solutions can instead be considered domain walls. Thus, in general, these wormhole solutions do not allow time travel.

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## I. INTRODUCTION

An important aspect in black hole physics is that they can be formed through gravitational collapse of matter. This is indeed the case for spherical collapse in an asymptotically flat background. For other backgrounds, such as asymptotically anti-de Sitter spacetime with plane symmetry, it was found that gravitational collapse of plane symmetric distributions of matter also results in an event horizon [1]. The event horizon may have planar [2], cylindrical [3] or toroidal topology [3, 4]. Indeed, the collapse of planar distributions of matter, in a background with a negative cosmological constant, can form a planar black hole (or a black membrane) violating somehow the hoop conjecture. Upon compactification of one or two coordinates one finds that cylindrical black holes (or black strings), or toroidal black holes can also form from the gravitational collapse of a cylindrical or toroidal distribution of matter, respectively. In these solutions the mass parameter is a surface mass energy in the planar case, a linear mass density in the cylindrical case, and a mass in the toroidal case [1].

A natural extension of these solutions would be to add exotic matter to obtain plane symmetric traversable wormhole solutions, with planar, cylindrical and toroidal topologies. These would add to other non-spherically symmetric wormholes which have already been considered by several authors. For instance, extending the spherically symmetric Morris-Thorne wormholes [5], Visser [6] motivated by the aim of minimizing the violation of the energy conditions and the possibility of a traveller not encountering regions of exotic matter in a traversal through a wormhole, constructed polyhedral wormholes and, in particular, cubic wormholes. These contained exotic matter concentrated only at the edges and the corners of the geometrical structure, and a traveller could pass through the flat faces without encountering matter, exotic or otherwise. González-Díaz generalized the static spherically symmetric traversable wormhole solution to that of a (non-planar) torus-like topology [7]. This geometrical construction was denoted as a ringhole. González-Díaz went on to analyze the causal structure of the solution, i.e., the presence of closed timelike

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curves, and has recently studied the ringhole evolution due to the accelerating expansion of the universe, in the presence of dark energy [8]. Other interesting non-spherically symmetric traversable spacetimes are the stationary solution obtained by Teo [9], and an axially symmetric traversable wormhole solution obtained by Kuhfittig [10]. In this work we study plane symmetric wormholes. We match an interior static and plane wormhole spacetime to a vacuum solution with a negative cosmological constant, i.e., to an anti-de Sitter spacetime. The properties of the junction surface, such as the surface stresses are determined. For the wormhole solutions quoted above, although the throat geometries differ from solution to solution, all these spacetimes are asymptotically flat with trivial topology, at infinity, whereas for the solutions we analyze the infinity carries the same topology as the throat, meaning that these solutions can also be considered as domain walls. Wormholes with this same property are the spherical wormholes joining two Friedmann-Robertson-Walker universes [11].

The plan of this paper is as follows: In Sec. II we present a plane symmetric metric with a negative cosmological constant and analyze the mathematics of embedding in order to obtain a wormhole solution. In Sec. III, we present the Einstein equations for the interior solution, and verify that the null energy condition is necessarily violated at the wormhole throat. In Sec. IV we deduce the exterior plane vacuum solution, through the Einstein equations. In Sec. V, we match an interior plane spacetime to the vacuum solution with a negative cosmological constant, and deduce the surface stresses at the thin-shell. By expressing the tangential surface pressure as a function of several parameters, namely, that of the matching radius, the radial derivative of the redshift function and of the surface energy density, the sign of the tangential surface pressure is analyzed. We also obtain a general equation governing the behavior of the radial pressure/tension across the junction in terms of the surface stresses. In section VI, we construct a plane traversable model that minimizes the null energy condition violation at the throat, and in which the surface stresses on the thin shell satisfy the energy conditions. Finally, we conclude in Sec. VII.

## II. THE METRIC AND THE EMBEDDING

### A. The metric

In this work we will be concerned with the possibility of the construction of traversable wormholes in static spacetimes and with plane symmetry, in the presence of a negative cosmological constant,  $\Lambda < 0$ . The plane symmetric metric will have the following general form

$$ds^2 = g_{tt}(r) dt^2 + g_{rr}(r) dr^2 + \alpha^2 r^2 (dx^2 + dy^2), \quad (1)$$

where  $\alpha$  is the inverse of the characteristic length of the system, which here we adopt as given by the negative cosmological constant, i.e., we put  $\alpha^2 = -\Lambda/3$ . The ranges of  $t$  and  $r$  are  $-\infty < t < +\infty$  and  $r_0 \leq r < +\infty$ , where  $r_0$  is the radius of the wormhole throat. The range of the coordinates  $x$  and  $y$  determine the topology of the plane symmetric metric. For the planar case, the topology of the two-dimensional space,  $t = \text{constant}$  and  $r = \text{constant}$ , is  $R^2$ , with coordinate range  $-\infty < x < +\infty$  and  $-\infty < y < +\infty$ . For the cylindrical case the topology is  $R \times S^1$ , with  $-\infty < x < +\infty$  and  $0 \leq \alpha y < 2\pi$ . For the toroidal case the topology is  $S^1 \times S^1$  (i.e., the torus  $T^2$ ), with  $0 \leq \alpha x < 2\pi$  and  $0 \leq \alpha y < 2\pi$  [3, 4, 12, 13].

We will adopt a specific form of a static and plane symmetric spacetime metric (with  $G = c = 1$ ), given by

$$ds^2 = -e^{2\Phi(r)} dt^2 + \left( \alpha^2 r^2 - \frac{m(r)}{\alpha r} \right)^{-1} dr^2 + \alpha^2 r^2 (dx^2 + dy^2), \quad (2)$$

where  $\Phi(r)$  and  $m(r)$  are arbitrary functions of the radial coordinate,  $r$ .  $\Phi(r)$  is called the redshift function, for it is related to the gravitational redshift. The redshift function will be considered finite throughout the spacetime, so that the appearance of event horizons is avoided.  $m(r)$  is called the form function, for it determines the shape of the wormhole, which can be visualized through embedding diagrams. We have kept the definitions of Morris and Thorne [5].

### B. The embedding

Embedding diagrams are a good tool to represent a wormhole and extract some useful information for the choice of the form function,  $m(r)$  [14]. For this plane symmetric wormhole one can use the treatment of embedding carried out by Morris and Thorne [5]. Consider the interior wormhole geometry at a fixed moment of time, and at fixed  $x$  and  $y$ . The metric is given by  $ds^2 = (\alpha^2 r^2 - m(r)/\alpha r)^{-1} dr^2$ . This interior can then be embedded in a two-dimensional Euclidean space,  $ds^2 = dz^2 + dr^2$ , where  $z$  is the new extra coordinate.

Here,  $z$  is only a function of  $r$ ,  $z = z(r)$ , and the condition for the embedding surface is then

$$\frac{dz}{dr} = \pm \left( \frac{1 - \alpha^2 r^2 + \frac{m(r)}{\alpha r}}{\alpha^2 r^2 - \frac{m(r)}{\alpha r}} \right)^{1/2}. \quad (3)$$

To be a solution of a wormhole, the geometry has a minimum radius,  $r = r_0$ , denoted as the throat, which defined in terms of the shape function is given by

$$m(r_0) = \alpha^3 r_0^3, \quad (4)$$

at which the embedded surface is vertical, i.e.,  $dz/dr \rightarrow \infty$ . As in a general wormhole solution, the radial coordinate  $r$  is ill-behaved near the throat, but the proper radial distance,  $l(r) = \pm \int_{r_0}^r (\alpha^2 r'^2 - m(r')/\alpha r')^{-1/2} dr'$ , is required to be finite throughout spacetime. This implies that the condition  $\alpha^2 r^2 - m(r)/\alpha r \geq 0$  is imposed. Moreover, the numerator inside the square root of Eq. (3) is in danger of going negative for sufficiently large  $r$ , so if one wants a full interior embedding the particular wormhole solution in study should take care of the problem. Now as for the exterior, the present solution of a plane symmetric wormhole is asymptotically anti-de Sitter. Equation (4) is still valid for the exterior but now  $m(r)$  is replaced by the total mass  $M$ . One then sees that this form of embedding is only valid for  $r < r_{\text{me}}$  where  $r_{\text{me}}$  gives the zero of the numerator inside the square root, i.e., the maximum embedding radius. This should not worry us. The importance of the embedding is near the throat region where a special condition, the flare out condition, should be obeyed. Of course, this treatment has the drawback of being highly coordinate dependent. For a covariant treatment see Hochberg and Visser [15, 16]. Indeed, to be a solution of a wormhole the imposition that the throat flares out is necessary. Mathematically, this flaring-out condition entails that the inverse of the embedding function,  $r(z)$ , must satisfy  $d^2r/dz^2 > 0$  at or near the throat,  $m(r_0) = \alpha^3 r_0^3$ . Differentiating  $dr/dz = \pm \left( \frac{\alpha^3 r^3 - m(r)}{\alpha r - \alpha^3 r^3 + m(r)} \right)^{1/2}$  with respect to  $z$ , we have

$$\frac{d^2r}{dz^2} = \frac{\alpha(2\alpha^3 r^3 - m'r + m)}{2(\alpha r - \alpha^3 r^3 + m)^2} > 0. \quad (5)$$

Equation (5) is the flaring-out condition, implying that at the throat,  $r = r_0$  or  $m(r_0) = \alpha^3 r_0^3$ , we have the important condition

$$m'(r_0) < 3\alpha^3 r_0^2, \quad (6)$$

which will play a fundamental role in the analysis of the violation of the energy conditions. Now one can draw an embedding diagram, where the function is plotted in a  $z \times r$  diagram, see Fig. 1.

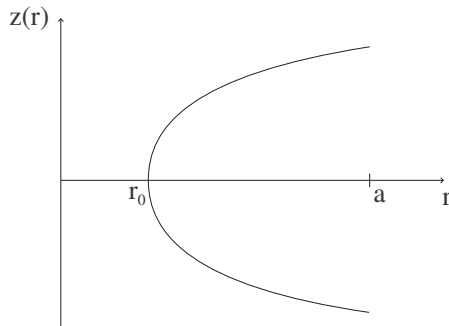


FIG. 1: Embedding diagram of the plane symmetric traversable wormhole. The throat is situated at  $r_0$ , and the exotic matter threading the wormhole extends to a junction radius,  $a$ , at which the interior solution is matched to an exterior vacuum solution. It is only possible to draw the planar topology in this embedding diagram.

Although the coordinate  $x$ , say, is not flat, one could think of extending the diagram in a perpendicular direction through this  $x$  coordinate. For the planar case this extends from  $-\infty$  to  $+\infty$ , and thus one can see that the construction of the wormhole does not involve a topology change. That is, the wormhole has trivial topology. In this sense, the wormhole solution can be thought of as a domain wall. Compactifying the other coordinate  $y$  does not alter the situation. To better visualize the three topologies we draw in Fig. 2 through pictorial diagrams, the planar, cylindrical and toroidal topologies in the  $r$  and  $x$  (or  $y$ ) plane.

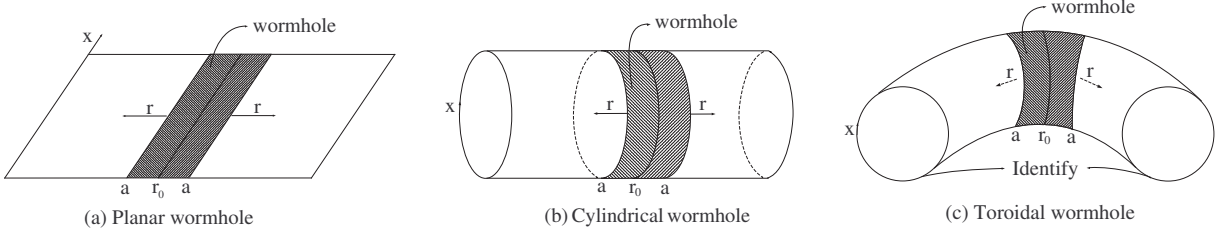


FIG. 2: Pictorial diagrams of the plane symmetric traversable wormhole. The throat is situated at  $r_0$ , and the exotic matter threading the wormhole extends to a junction radius,  $a$ , at which the interior solution is matched to an exterior vacuum solution. In figure (a), we have the planar case, which can also be considered as a domain wall connecting two universes at the throat. In figure (b), one of the coordinates has been compactified, resulting in a cylindrical topology. In figure (c), both coordinates have been compactified, resulting in a planar toroidal topology.

From Fig. 2 it is clear that the construction of these wormholes does not alter the spatial topology of the background spacetime (i.e., spacetime is not multiply-connected), and the domain wall interpretation can be easily recognized. This feature appears also in the spherical wormholes joining two Friedmann-Robertson-Walker universes [11]. Note that some authors [17] do not consider as wormholes, solutions that do not alter the topology of space. In general, such solutions do not allow time travel.

### III. EINSTEIN EQUATIONS WITH $\Lambda < 0$ : INTERIOR SOLUTION

#### A. The equations

Consider an orthonormal reference frame, i.e., the proper reference frame of a set of observers who remain at rest in the coordinate system, with  $(r, x, y)$  fixed. In the orthonormal basis the only non-zero components of the stress-energy tensor,  $T_{\hat{\mu}\hat{\nu}}$ , are  $T_{\hat{t}\hat{t}}$ ,  $T_{\hat{r}\hat{r}}$ , and  $T_{\hat{x}\hat{x}} = T_{\hat{y}\hat{y}}$ . Written in a diagonal form,  $T_{\hat{\mu}\hat{\nu}} = \text{diag}[\rho(r), p_r(r), p_x(r), p_y(r)]$ , the physical interpretation of the respective components is:  $\rho(r)$  is the energy density;  $p_r(r)$  is the radial pressure; and  $p_x(r)$  and  $p_y(r)$  are the tangential pressures measured in the  $\partial_x$  and  $\partial_y$  directions, respectively. We define  $p(r) = p_x(r) = p_y(r)$ .

In the presence of a non-vanishing cosmological constant the Einstein field equation is given by  $G_{\hat{\mu}\hat{\nu}} + \Lambda g_{\hat{\mu}\hat{\nu}} = 8\pi T_{\hat{\mu}\hat{\nu}}$ . Considering a negative cosmological constant, we shall substitute  $\Lambda = -3\alpha^2$  in the analysis that follows. Using the metric, Eq. (2), we obtain from the Einstein field equation the following set of equations for  $\rho(r)$ ,  $p_r(r)$  and  $p(r)$ ,

$$\rho(r) = \frac{1}{8\pi} \frac{m'}{\alpha r^2}, \quad (7)$$

$$p_r(r) = \frac{1}{8\pi} \left[ \left( \alpha^2 r^2 - \frac{m}{\alpha r} \right) \left( \frac{1}{r^2} + 2 \frac{\Phi'}{r} \right) - 3\alpha^2 \right], \quad (8)$$

$$p(r) = \frac{1}{8\pi} \left\{ \left( \alpha^2 r^2 - \frac{m}{\alpha r} \right) \left[ \Phi'' + (\Phi')^2 + \frac{2\alpha^3 r^3 - m'r + m}{2r(\alpha^3 r^3 - m)} \Phi' \right. \right. \\ \left. \left. + \frac{2\alpha^3 r^3 - m'r + m}{2r^2(\alpha^3 r^3 - m)} + \frac{\Phi'}{r} \right] - 3\alpha^2 \right\}. \quad (9)$$

By taking the derivative with respect to the radial coordinate,  $r$ , of Eq. (8), and eliminating  $m'$  and  $\Phi''$ , given in Eq. (7) and Eq. (9), respectively, we obtain the following equation

$$p'_r = -(\rho c^2 + p_r)\Phi' - \frac{2}{r}(p - p_r), \quad (10)$$

which can also be obtained using the conservation of the stress-energy tensor,  $T^{\hat{\mu}\hat{\nu}}_{;\hat{\nu}} = 0$ , setting  $\hat{\mu} = r$ . Equation (10) can be interpreted as the relativistic Euler equation, or the hydrostatic equation for equilibrium for the material threading the wormhole.

## B. The energy conditions for the interior

We verify that the condition of Eq. (6) entails the violation of the null energy condition (NEC) at the throat and of the weak energy condition (WEC) in a specific domain determined below [5, 18, 19]. Explicitly the WEC states  $T_{\mu\nu}U^\mu U^\nu \geq 0$ , i.e.,  $\rho(r) \geq 0$  and  $\rho(r) + p_r(r) \geq 0$ , where  $U^\mu$  is a timelike vector and  $T_{\mu\nu}$  is the stress-energy tensor. Its physical interpretation is that the local energy density is positive. In the orthonormal frame, in the throat vicinity, we have

$$\rho(r_0) = T_{\hat{\mu}\hat{\nu}}U^{\hat{\mu}}U^{\hat{\nu}} = \frac{1}{8\pi} \frac{m'(r_0)}{\alpha r_0^2}, \quad (11)$$

The condition of Eq. (6) implies  $\rho(r_0) \geq 0$  for  $0 \leq m'(r_0) < 3\alpha^3 r_0^2$ , while  $\rho(r_0) < 0$  for  $m'(r_0) > 3\alpha^3 r_0^2$ .

By continuity the NEC can be obtained from the WEC. The NEC states that  $T_{\mu\nu}k^\mu k^\nu \geq 0$ , i.e.,  $\rho(r) + p_r(r) \geq 0$ , where  $k^\mu$  is a null vector. In the orthonormal frame,  $k^{\hat{\mu}} = (1, 1, 0, 0)$ , we have

$$T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} = \rho(r) + p_r(r) = \frac{1}{8\pi} \left[ \frac{m' - 3\alpha^3 r^2}{\alpha r^2} + \left( \alpha^2 r^2 - \frac{m}{\alpha r} \right) \left( \frac{1}{r^2} + 2\frac{\Phi'}{r} \right) \right], \quad (12)$$

which at the throat,  $m(r_0) = \alpha^3 r_0^3$ , due to the finiteness of  $\Phi'$ , reduces to

$$T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} = \frac{1}{8\pi} \frac{m'(r_0) - 3\alpha^3 r_0^2}{\alpha r_0^2}. \quad (13)$$

Equation (6) implies the violation of the NEC at the wormhole throat, i.e.,  $T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} < 0$ . Matter that violates the NEC is denoted as exotic matter.

## IV. EINSTEIN EQUATIONS WITH $\Lambda < 0$ : EXTERIOR VACUUM SOLUTION

In general, the solutions of the interior and exterior spacetimes are given in different coordinate systems. Therefore, to distinguish between both spacetimes we shall write out the interior and exterior metrics in the following coordinate systems,  $(t, r, x, y)$  and  $(\bar{t}, \bar{r}, \bar{x}, \bar{y})$ , respectively. We consider the most general case in which the interior and exterior cosmological constants,  $\Lambda$  and  $\bar{\Lambda}$ , are different. Therefore, the characteristic lengths are defined as  $\alpha^{-1} = \sqrt{-3/\Lambda}$  and  $\bar{\alpha}^{-1} = \sqrt{-3/\bar{\Lambda}}$ , respectively. The spacetime geometry for a vacuum exterior region is simply determined considering a null stress-energy tensor,  $T_{\hat{\mu}\hat{\nu}} = 0$ , i.e.,  $\bar{\rho}(\bar{r}) = \bar{p}_r(\bar{r}) = \bar{p}(\bar{r}) = 0$ . Thus, the Einstein equations, Eqs. (7)-(9), written in the coordinate system,  $(\bar{t}, \bar{r}, \bar{x}, \bar{y})$ , with  $\bar{m} = M$  and  $e^{2\bar{\Phi}(\bar{r})} = (\bar{\alpha}^2 \bar{r}^2 - M/\bar{\alpha}\bar{r})$  provide us with the exterior vacuum solution with plane symmetry and a negative cosmological constant, given by

$$d\bar{s}^2 = - \left( \bar{\alpha}^2 \bar{r}^2 - \frac{M}{\bar{\alpha}\bar{r}} \right) d\bar{t}^2 + \left( \bar{\alpha}^2 \bar{r}^2 - \frac{M}{\bar{\alpha}\bar{r}} \right)^{-1} d\bar{r}^2 + \bar{\alpha}^2 \bar{r}^2 (d\bar{x}^2 + d\bar{y}^2). \quad (14)$$

$M$  is a constant of integration and is a mass parameter [13]. Equation (14) is the metric of a black membrane, black string, or toroidal black hole, when the metric possesses an event horizon. Considering a positive value for  $M$ , an event horizon occurs at  $\bar{r}_h = M^{1/3}/\bar{\alpha}$ . The scalar Kretschmann polynomial is given by

$$R^{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}R_{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}} = 24\bar{\alpha}^4 + 12\frac{M^2}{\bar{\alpha}^2 \bar{r}^6}. \quad (15)$$

showing that a singularity occurs at  $\bar{r} = 0$ . But this is no problem to us, since we are interested in solutions not containing black holes.

## V. JUNCTION CONDITIONS FOR PLANE SYMMETRIC WORMHOLES WITH AN EXTERIOR $\Lambda < 0$ VACUUM

### A. Matching of the equations I: The surface stresses

#### 1. The surface stresses

We shall match the interior spacetime given by Eq. (2) to an exterior vacuum solution with a negative cosmological constant, given by Eq. (14), at a junction boundary  $\Sigma$ , which is situated at  $r = \bar{r} = a$ . The intrinsic metric to  $\Sigma$  is

given by

$$ds_\Sigma^2 = -d\tau^2 + \alpha^2 a^2 (dx^2 + dy^2), \quad (16)$$

where  $\tau$  is the proper time on  $\Sigma$ . Note that the junction surface is situated outside the event horizon, i.e.,  $a > \bar{r}_b = M^{1/3}/\bar{\alpha}$ , to avoid a black hole solution.

The surface stresses at the thin shell  $\Sigma$  can be determined from the discontinuities in the extrinsic curvatures,  $K_{ij}$  (see Appendix), and are given by

$$\sigma = -\frac{1}{4\pi a} \left( \sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}} - \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}} \right), \quad (17)$$

$$\mathcal{P} = \frac{1}{8\pi a} \left( \frac{2\bar{\alpha}^2 a^2 - \frac{M}{2\bar{\alpha}a}}{\sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}}} - \zeta \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}} \right), \quad (18)$$

where  $\sigma$  is the surface energy density, and  $\mathcal{P}$ , the tangential surface pressure on  $\Sigma$ . We have defined  $\zeta = 1 + a\Phi'(a)$  for notational convenience. In the analysis that follows, we will only be interested in the  $M > 0$  case. We shall, in general, consider a non-zero redshift function.

A particularly simple solution is given when  $(\alpha^2 a^2 - m(a)/\alpha a) = (\bar{\alpha}^2 a^2 - M/\bar{\alpha}a)$ , in which case Eqs. (17)-(18) reduce to

$$\sigma_0 = 0, \quad (19)$$

$$\mathcal{P}_0 = \frac{1}{8\pi a} \frac{(2 - \zeta)\bar{\alpha}^2 a^2 + (\zeta - \frac{1}{2}) \frac{M}{\bar{\alpha}a}}{\sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}}}. \quad (20)$$

For a similar analysis in the spherically symmetric case, see [20].

## 2. The energy conditions at the junction

In general, the violation of the energy conditions of the surface stresses on the boundary is verified. However, one obtains interesting restrictions in imposing the energy conditions on the junction surface,  $\Sigma$ . The WEC implies  $\sigma \geq 0$  and  $\sigma + \mathcal{P} \geq 0$ , and by continuity implies the NEC, i.e.,  $\sigma + \mathcal{P} \geq 0$ .

From Eqs. (17)-(18), we deduce

$$\sigma + \mathcal{P} = \frac{1}{8\pi a} \left[ (2 - \zeta) \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}} + \frac{\frac{3M}{2\bar{\alpha}a}}{\sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}}} \right]. \quad (21)$$

By imposing a non-negative surface energy density,  $\sigma \geq 0$ , the NEC is satisfied, i.e.,  $\sigma + \mathcal{P} \geq 0$ , if the following condition is verified,

$$\zeta \leq \frac{2\bar{\alpha}^2 a^2 - \frac{M}{2\bar{\alpha}a}}{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}}. \quad (22)$$

## 3. Special cases

Taking into account Eqs. (17)-(18), one may express  $\mathcal{P}$  as a function of  $\sigma$ , by

$$\mathcal{P} = \frac{1}{8\pi a} \left[ \frac{(2 - \zeta)\bar{\alpha}^2 a^2 + (\zeta - \frac{1}{2}) \frac{M}{\bar{\alpha}a}}{\sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}}} - 4\pi a \zeta \sigma \right]. \quad (23)$$

To analyze Eq. (23), namely, to find domains in which  $\mathcal{P}$  assumes the nature of a tangential surface pressure,  $\mathcal{P} > 0$ , or a tangential surface tension,  $\mathcal{P} < 0$ , it is convenient to express Eq. (23) in the following compact form

$$\mathcal{P} = \frac{\bar{\alpha}}{8\pi} \frac{\Gamma(\xi, \zeta, \mu)}{\sqrt{1 - \xi^3}}, \quad (24)$$

with  $\xi = M^{1/3}/(\bar{\alpha}a)$  and  $\mu = 4\pi\sigma/\bar{\alpha}$ .  $\Gamma(\xi, \zeta, \mu)$  is defined as

$$\Gamma(\xi, \zeta, \mu) = (2 - \zeta) + \left(\zeta - \frac{1}{2}\right) \xi^3 - \mu\zeta\sqrt{1 - \xi^3}. \quad (25)$$

One may now fix one of the parameters and analyze the sign of  $\Gamma(\xi, \zeta, \mu)$ , and consequently the sign of  $\mathcal{P}$ .

The cases we shall analyze are: firstly, that of a zero surface energy density,  $\sigma = 0$ , i.e.,  $\mu = 0$ ; secondly, a constant redshift function,  $\Phi'(r) = 0$ , i.e.,  $\zeta = 1$ ; and finally, we will consider two specific equations of state. The first is that of a domain wall [19, 21], which is a hypersurface with an intrinsic equation of state of the form  $\sigma + \mathcal{P} = 0$ . This equation of state is of particular interest in the present analysis, for planar traversable wormholes may be visualized as domain walls connecting different universes. The second equation of state we shall consider is that of the traceless surface stress-energy tensor,  $S^i_i = 0$ , i.e.,  $-\sigma + 2\mathcal{P} = 0$ , which is an equation of state provided by the Casimir effect with a massless field [19]. The Casimir effect is frequently invoked to provide exotic matter to a system violating the energy conditions.

*(i) Null surface energy density*

Now, considering a zero surface energy density,  $\sigma = 0$ , i.e.,  $\mu = 0$ , Eq. (25) reduces to

$$\Gamma(\xi, \zeta) = (2 - \zeta) + \left(\zeta - \frac{1}{2}\right) \xi^3, \quad (26)$$

which is analyzed in Fig. 3. Qualitatively, we verify that for low values of  $\xi$  (high  $a$ ) and high values of  $\zeta$ ,  $\Gamma(\xi, \zeta)$  is negative, implying a surface tension. For low values of  $\zeta$  and for all  $\xi$ ,  $\Gamma(\xi, \zeta)$  is positive, implying a surface pressure. For high values of  $\xi$  (low  $a$ , in the proximity of  $r_b = M^{1/3}/\bar{\alpha}$ ) and for all  $\zeta$ , a surface pressure is needed to hold the structure against collapse.

We verify that a surface boundary,  $\mathcal{P} = 0$ , i.e.,  $\Gamma(\xi, \zeta) = 0$ , is obtained, if  $\zeta_0 = (2 - \xi^3/2)/(1 - \xi^3)$ . This condition is equivalent to  $\Phi'(a) = (\bar{\alpha}^2 a + \frac{M}{2a^2})/(\bar{\alpha}^2 a^2 - \frac{M}{a})$ . For a tangential surface pressure,  $\Gamma(\xi, \zeta) > 0$ , we have  $\zeta < \zeta_0$ . For a tangential surface tension,  $\Gamma(\xi, \zeta) < 0$ , the condition  $\zeta > \zeta_0$  is imposed. In particular, considering a constant redshift function, i.e.,  $\Phi'(r) = 0$ , so that  $\zeta = 1$ , we verify that  $\mathcal{P}$  is a tangential surface pressure,  $\mathcal{P} > 0$ .

For  $\sigma = 0$ , we verify that the WEC is satisfied only if  $\mathcal{P} \geq 0$ . This condition is verified in the domain  $\zeta \leq \zeta_0$ , which is consistent with Eq. (22).

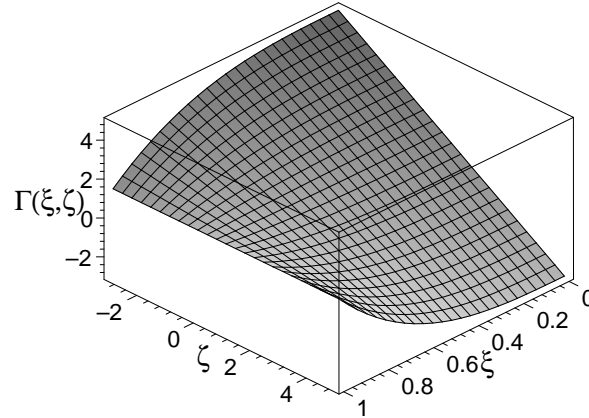


FIG. 3: Considering a zero surface energy density,  $\sigma = 0$ , i.e.,  $\mu = 0$ , we have defined  $\xi = M^{1/3}/(\bar{\alpha}a)$  and  $\zeta = 1 + a\Phi'(a)$ . The surface is defined by  $\Gamma(\xi, \zeta) = (2 - \zeta) + (\zeta - 1/2)\xi^3$ . One verifies that for low values of  $\xi$  (high  $a$ ) and high values of  $\zeta$ ,  $\Gamma(\xi, \zeta)$  is negative, implying a surface tension,  $\mathcal{P} < 0$ . Whilst for low values of  $\zeta$  and for all  $\xi$ ,  $\Gamma(\xi, \zeta)$  is positive, implying a surface pressure,  $\mathcal{P} > 0$ . For high values of  $\xi$  (low  $a$ , in the proximity of  $r_b$ ) and for arbitrary values of  $\zeta$ , a surface pressure is needed to hold the structure against collapse. See text for details.

*(ii) Constant redshift function*

Considering the specific case of a constant redshift function,  $\Phi'(r) = 0$ , i.e.,  $\zeta = 1$ , Eq. (25) reduces to

$$\Gamma(\xi, \mu) = 1 + \frac{\xi^3}{2} - \mu\sqrt{1 - \xi^3}. \quad (27)$$

This relationship is represented as the surface in Fig. 4. Qualitatively, considering low values of  $\xi$  (high  $a$ ) and high positive values of  $\mu$ ,  $\mathcal{P}$  assumes the character of a surface tension, i.e.,  $\mathcal{P} < 0$ . For negative values of the surface energy density and for all  $\xi$ ,  $\mathcal{P}$  is always positive. For extremely high values of  $\xi$  (extremely low  $a$ , in the proximity of  $r_b = M^{1/3}/\bar{\alpha}$ ) and for all  $\sigma$ , a surface pressure is needed to hold against collapse.

Analytically, we verify that  $\mathcal{P} = 0$  at  $\mu_0 = (1 + \xi^3/2)/\sqrt{1 - \xi^3}$ ;  $\mathcal{P}$  assumes a tangential surface pressure,  $\mathcal{P} > 0$ , for  $\mu < \mu_0$ , and a tangential surface tension,  $\mathcal{P} < 0$ , for  $\mu > \mu_0$ .

For  $\zeta = 1$ , i.e.,  $\Phi'(r) = 0$ , by imposing  $\sigma \geq 0$ , we verify that by taking into account Eq. (21), the WEC is immediately satisfied. If  $\sigma < 0$ , then only the WEC is violated, while the NEC is satisfied.

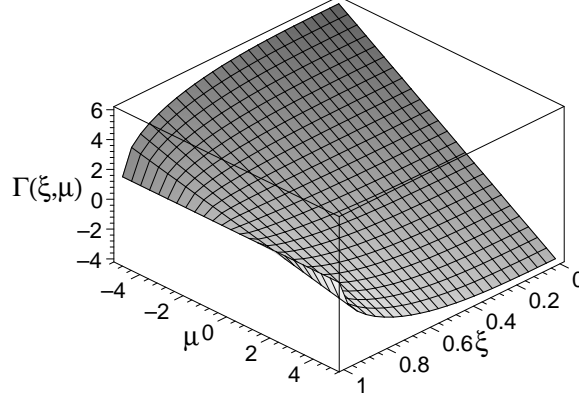


FIG. 4: Considering a constant value for the redshift function,  $\Phi'(r) = 0$ , i.e.,  $\zeta = 1$ , we have defined  $\xi = M^{1/3}/(\bar{\alpha}a)$  and  $\mu = 4\pi\sigma/\bar{\alpha}$ . The surface is defined by  $\Gamma(\xi, \mu) = 1 + \xi^3/2 - \mu\sqrt{1 - \xi^3}$ . For low values of  $\xi$  (high  $a$ ) and positive values of  $\mu$ ,  $\Gamma(\xi, \mu)$  is negative, implying a surface tension,  $\mathcal{P} < 0$ . While, for negative values of the surface energy density and for all  $\xi$ ,  $\Gamma(\xi, \mu)$  is positive, implying a surface pressure,  $\mathcal{P} > 0$ . For extremely high values of  $\xi$  (for values of  $a$  in the neighborhood of  $r_b$ ), a surface pressure is once again required to hold against collapse. See text for details.

### (iii) Domain wall

The domain wall is a hypersurface with an intrinsic equation of state of the form  $\sigma + \mathcal{P} = 0$  [19, 21]. Taking into account Eqs. (17)-(18), from  $\sigma + \mathcal{P} = 0$ , we deduce the following condition

$$(\zeta - 2)\sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}} = \frac{\frac{3M}{2\bar{\alpha}a}}{\sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}}}. \quad (28)$$

The right hand side term is always positive, for by construction we have only taken into account  $M > 0$ . Thus, to have a solution, this imposes the important condition  $\zeta > 2$ .

The tangential surface pressure is given by

$$\mathcal{P} = \frac{1}{4\pi a} \frac{1}{\sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}}} \left[ \bar{\alpha}^2 a^2 - \left( \frac{2\zeta - 1}{\zeta - 2} \right) \frac{M}{2\bar{\alpha}a} \right]. \quad (29)$$

Equation (29) may also be obtained by substituting  $\sigma = -\mathcal{P}$  into Eq. (23), and can be written in a compact form as  $\mathcal{P} = \frac{\bar{\alpha}}{4\pi} \frac{\Gamma(\xi, \zeta)}{\sqrt{1 - \xi^3}}$ , with  $\Gamma(\xi, \zeta)$  given by

$$\Gamma(\xi, \zeta) = 1 - \left( \frac{2\zeta - 1}{\zeta - 2} \right) \frac{\xi^3}{2}. \quad (30)$$

$\Gamma(\xi, \zeta)$  is represented as a surface in Fig. 5. Qualitatively, we verify that for low values of  $\xi$  (high  $a$ ) and for all  $\zeta$ ,  $\Gamma(\xi, \zeta)$  is positive, implying a surface pressure,  $\mathcal{P} > 0$ . While, for high values of  $\xi$  (low  $a$ ) and for all  $\zeta$ ,  $\Gamma(\xi, \zeta)$  is negative, implying a surface tension,  $\mathcal{P} < 0$ . We have a null tangential surface pressure,  $\mathcal{P} = 0$ , for  $\zeta_0 = (2 - \xi^3/2)/(1 - \xi^3)$ .  $\mathcal{P}$  assumes the character of a surface pressure,  $\mathcal{P} > 0$ , for  $\zeta > \zeta_0$  and a surface tension,  $\mathcal{P} < 0$ , for  $2 < \zeta < \zeta_0$ .

From the equation of state,  $\sigma = -\mathcal{P}$ , we verify that  $\sigma \geq 0$  if and only if  $\mathcal{P} \leq 0$ , thus satisfying the WEC.



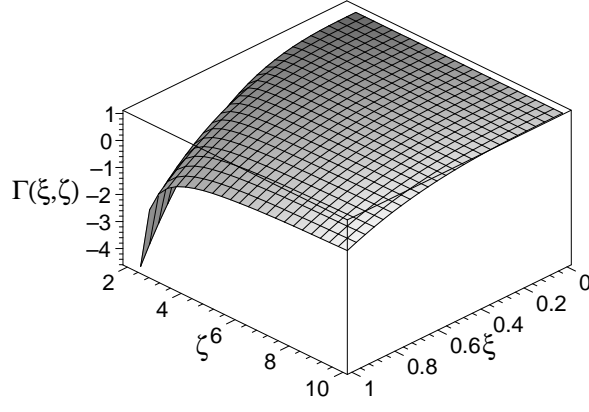


FIG. 5: The surface represents the sign of  $\mathcal{P}$  for a domain wall, with the equation of state  $\sigma + \mathcal{P} = 0$ . We have defined  $\xi = M^{1/3}/(\bar{\alpha}a)$  and  $\zeta = 1 + a\Phi'(a)$ . For low values of  $\xi$  (high  $a$ ) and for all  $\zeta$ ,  $\Gamma(\xi, \zeta)$  is positive, implying a surface pressure,  $\mathcal{P} > 0$ . While, for high values of  $\xi$  (low  $a$ ) and for all  $\zeta$ ,  $\Gamma(\xi, \zeta)$  is negative, implying a surface tension,  $\mathcal{P} < 0$ . See text for details.

(iv) *Traceless surface stress-energy tensor*

The traceless surface stress-energy tensor,  $S^i_i = 0$ , i.e.,  $-\sigma + 2\mathcal{P} = 0$ , is a case of particular interest [19]. The Casimir effect with a massless field provides one with a stress-energy tensor of this type. Usually, the Casimir effect is theoretically invoked to provide exotic matter to the system considered at hand. From  $\sigma = 2\mathcal{P}$  and taking into account Eqs. (17)-(18), we have the following condition

$$(1 + \zeta) \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}} = \frac{3 \left( \bar{\alpha}^2 a^2 - \frac{M}{2\bar{\alpha}a} \right)}{\sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}}}. \quad (31)$$

The right hand side term is always positive, therefore we need to impose the condition  $\zeta > -1$  to have a solution. The tangential surface pressure is given by

$$\mathcal{P} = \frac{1}{8\pi a} \frac{1}{\sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}}} \left[ \left( \frac{2 - \zeta}{1 + \zeta} \right) \bar{\alpha}^2 a^2 - \left( \frac{1 - 2\zeta}{1 + \zeta} \right) \frac{M}{2\bar{\alpha}a} \right]. \quad (32)$$

Equation (32) may also be obtained by substituting  $\sigma = 2\mathcal{P}$  into Eq. (23), and can be written in a compact form as  $\mathcal{P} = \frac{\bar{\alpha}}{8\pi} \frac{\Gamma(\xi, \zeta)}{\sqrt{1 - \xi^3}}$ , with  $\Gamma(\xi, \zeta)$  given by

$$\Gamma(\xi, \zeta) = \left( \frac{2 - \zeta}{1 + \zeta} \right) - \left( \frac{1 - 2\zeta}{1 + \zeta} \right) \frac{\xi^3}{2}. \quad (33)$$

Qualitatively, from Fig. 6, we verify that for low values of  $\xi$  (high  $a$ ) and high values of  $\zeta$ ,  $\Gamma(\xi, \zeta)$  is negative, implying a surface tension,  $\mathcal{P} < 0$ . While, for high values of  $\xi$  (low  $a$ , in the proximity of  $r_b$ ) and for all  $\zeta$ ,  $\Gamma(\xi, \zeta)$  is positive, implying a surface pressure,  $\mathcal{P} > 0$ . In addition, one also verifies that for low values of  $\zeta$  and for all  $\xi$ ,  $\Gamma(\xi, \zeta)$  is also positive, implying a surface pressure. We have  $\mathcal{P} = 0$ , i.e., a surface boundary (as  $\sigma = 0$  from the equation of state  $\sigma = 2\mathcal{P}$ ), for  $\zeta_0 = (2 - \xi^3/2)/(1 - \xi^3)$ ; a tangential surface pressure,  $\mathcal{P} > 0$ , for  $-1 < \zeta < \zeta_0$  and a tangential surface tension,  $\mathcal{P} < 0$ , for  $\zeta > \zeta_0$ .

In relation to the energy conditions, from the equation of state,  $\sigma = 2\mathcal{P}$ , one readily verifies that  $\sigma \geq 0$ , if  $\mathcal{P} \geq 0$ . Thus, the WEC is satisfied only if  $\mathcal{P} \geq 0$ . This restriction is verified in the domain  $-1 < \zeta \leq \zeta_0$ , which is consistent with Eq. (22).

## B. Matching of the equations II: The radial pressure

To construct specific solutions of planar wormholes with a generic cosmological constant, the behavior of the radial pressure across the junction surface  $\Sigma$  is required. The pressure balance equation which relates the radial pressure

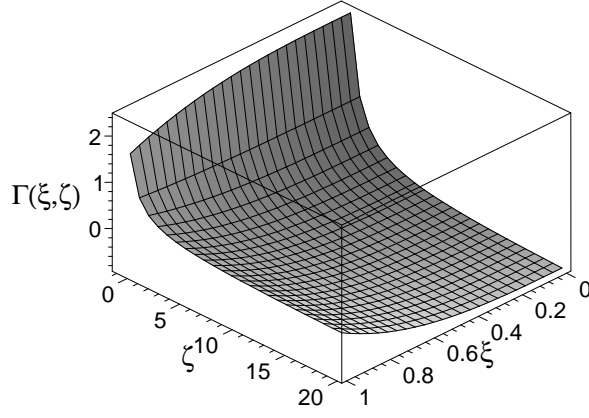


FIG. 6: The graph represents the sign of  $\mathcal{P}$  for the traceless stress-energy tensor, with the equation of state  $\sigma - 2\mathcal{P} = 0$ . We have defined  $\xi = M^{1/3}/(\bar{\alpha}a)$  and  $\zeta = 1 + a\Phi'(a)$ . For low values of  $\xi$  (high  $a$ ) and high values of  $\zeta$ ,  $\Gamma(\xi, \zeta)$  is negative, implying a surface tension,  $\mathcal{P} < 0$ . While, for high values of  $\xi$  (low  $a$ , in the proximity of  $r_b$ ) and for all  $\zeta$ ,  $\Gamma(\xi, \zeta)$  is positive, implying a surface pressure,  $\mathcal{P} > 0$ . In addition, one also verifies that for low values of  $\zeta$  and for all  $\xi$ ,  $\Gamma(\xi, \zeta)$  is also positive, implying a surface pressure. See text for details.

across the boundary in terms of the surface stresses of the thin shell [19], deduced in the Appendix, is given by

$$\begin{aligned} \left( \bar{p}_r(a) + \frac{3\bar{\alpha}^2}{8\pi} \right) - \left( p_r(a) + \frac{3\alpha^2}{8\pi} \right) &= \frac{1}{a} \left( \sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}} + \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}} \right) \mathcal{P} \\ &\quad - \frac{1}{2} \left( \frac{\bar{\alpha}^2 a + \frac{M}{2\bar{\alpha}a^2}}{\sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}}} + \Phi'(a) \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}} \right) \sigma, \end{aligned} \quad (34)$$

where  $\sigma$  and  $\mathcal{P}$  are given by Eqs. (17)-(18), respectively. Since  $\bar{p}_r(\bar{r}) = 0$  Eq. (34) is reduced to

$$\begin{aligned} p_r(a) &= \frac{3}{8\pi} (\bar{\alpha}^2 - \alpha^2) - \frac{1}{a} \left( \sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}} + \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}} \right) \mathcal{P} \\ &\quad + \frac{1}{2} \left( \frac{\bar{\alpha}^2 a + \frac{M}{2\bar{\alpha}a^2}}{\sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}}} + \Phi'(a) \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}} \right) \sigma, \end{aligned} \quad (35)$$

giving the interior radial pressure at the boundary as a function of the metric and matter fields at the junction surface.

## VI. MODEL OF A PLANAR TRAVERSABLE WORMHOLE MINIMIZING THE USAGE OF EXOTIC MATTER

One can also construct planar traversable wormhole solutions that minimize the region of the null energy condition violation. As the violation of the energy conditions is a particularly problematic issue, depending on one's point of view, it is of particular interest to minimize the exotic matter considered at hand [22, 23, 24]. For instance, one can confine the exotic matter to an arbitrarily small region around the throat, and consider that the surface stresses on the thin shell obey the energy conditions. The approach of Kuhfittig [23, 24], which we will follow, is to choose a special form function  $m(r)$  that can be made to minimize the violation of the NEC. This yields interestingly enough results for our planar case, as it has yielded for the spherically symmetric case presented in [23].

The interior wormhole solution extends from the throat at  $r_0$ , to a radius  $a$ , where it is matched to an exterior vacuum solution. To minimize the null energy condition violation, we wish to deduce a form function,  $m(r)$ , that somehow yields a solution arbitrarily close to a solution of  $\rho(r) + p_r(r) = 0$ . To simplify the analysis, consider a constant redshift function,  $\Phi'(r) = 0$ . Taking into account Eq. (12) with  $\rho(r) + p_r(r) = 0$ , we deduce  $m(r) = \alpha^3 r^3 - k\alpha r$ , with  $k > 0$ . With this motivation, we define the following form function

$$m(r) = \alpha^3 r^3 - k\alpha r + \alpha \epsilon(r), \quad \text{for } r_0 \leq r \leq a, \quad (36)$$

where  $\epsilon$  is a small positive adjustment in the interval  $r_0 \leq r \leq a$ . To satisfy  $m(r_0) = \alpha^3 r_0^3$  at the throat, the condition  $\epsilon(r_0) = kr_0$  is imposed. With the definition of Eq. (36), and considering the particular case of  $\epsilon'(r_0) = 0$ , we also verify that the flaring-out condition, Eq. (6), is met if  $k > 0$ .

In the interval  $r_0 \leq r \leq a$ , from Eqs. (7)-(9) we have the following stress-energy functions

$$\rho(r) = \frac{1}{8\pi} \left( -\frac{k - \epsilon'(r)}{r^2} + 3\alpha^2 \right), \quad (37)$$

$$p_r(r) = \frac{1}{8\pi} \left( \frac{k - \epsilon(r)/r}{r^2} - 3\alpha^2 \right), \quad (38)$$

$$p(r) = \frac{1}{8\pi} \left( \frac{\epsilon(r) - r\epsilon'(r)}{2r^3} - 3\alpha^2 \right). \quad (39)$$

The addition of Eqs. (37) and (38) entails the following relationship

$$\rho(r) + p_r(r) = -\frac{1}{8\pi} \frac{\epsilon(r) - \epsilon'(r)r}{r^3}. \quad (40)$$

One may now consider  $\epsilon'(r_0) = 0$ . An example satisfying this condition is

$$\epsilon(r) = kr_0 [1 - (r - r_0)^2 / (a - r_0)^2], \quad (41)$$

(see [23] for a similar example). Thus, Eq. (40) at the throat, taking into account  $\epsilon(r_0) = kr_0$  with  $k > 0$ , reduces to

$$\rho(r_0) + p_r(r_0) = -\frac{1}{8\pi} \frac{k}{r_0^2}. \quad (42)$$

Note that  $k$  can be made arbitrarily close to zero, and consequently, the region of the null energy condition violation can be made arbitrarily small.

Now we have to check that the surface stresses are in line with the idea of minimizing the NEC. Matching Eq. (2) with the exterior vacuum solution, Eq. (14), provides the surface stresses of Eqs. (17)-(18). From Eq. (21), with  $\zeta = 1$  as  $\Phi'(r) = 0$ , we verify that the NEC is always verified, i.e.,

$$\sigma + \mathcal{P} = \frac{1}{8\pi a} \left( \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}} + \frac{\frac{3M}{2\bar{\alpha}a}}{\sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}}} \right). \quad (43)$$

Here, we consider a positive mass parameter,  $M > 0$ , so that we have  $\sigma + \mathcal{P} > 0$ , satisfying the NEC. We can go further and impose the WEC. In this case, we have  $\sigma \geq 0$ , i.e.,  $(\alpha^2 a^2 - m(a)/\alpha a) \geq (\bar{\alpha}^2 a^2 - M/\bar{\alpha}a)$ . Taking into account the specific form function of Eq. (36), one deduces a restriction for the mass (per unit area) given by  $M \geq \bar{\alpha}^3 a^3 - \bar{\alpha}ak + \bar{\alpha}\epsilon(a)$ . Considering the specific choice of Eq. (41), we verify that  $\epsilon(a) = 0$  at the junction, so that the restriction imposed on the mass parameter reduces to  $M \geq \bar{\alpha}^3 a^3 - \bar{\alpha}ak$ . Thus, the WEC holds for  $M > \bar{\alpha}^3 a^3$ .

In particular, considering a zero surface energy density,  $\sigma = 0$ , the surface stresses on the thin shell, Eqs. (19)-(20), with  $\zeta = 1$ , reduce to

$$\sigma_0 = 0, \quad (44)$$

$$\mathcal{P}_0 = \frac{1}{8\pi a} \frac{\bar{\alpha}^2 a^2 + \frac{M}{2\bar{\alpha}a}}{\sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}}}. \quad (45)$$

Thus  $\mathcal{P}$  is always positive  $\mathcal{P} > 0$ , i.e., a surface pressure, satisfying the WEC and NEC. For  $\sigma = 0$ , considering Eq. (41), the mass (per unit area) reduces to  $M = \bar{\alpha}^3 a^3 - \bar{\alpha}ak$ . Note, however, that as  $k \rightarrow 0$ ,  $M$  can be arbitrarily close to  $\bar{\alpha}^3 a^3$  but not equal, as from the equality case we get an unwanted event horizon.

## VII. CONCLUSION

We have constructed plane symmetric wormholes (with planar, cylindrical and toroidal topologies) in an anti-de Sitter background. We have determined the surface stresses, analyzed the sign of the tangential surface pressure, displayed an equation relating the radial pressure across the junction boundary, and given a model which minimizes the usage of exotic matter. We have found that the construction of these wormholes does not involve a topology change, and thus the wormhole solution can be considered a domain wall. As such these wormholes do not allow time travel. We have not considered in this work solutions with zero or positive cosmological constants as, for plane symmetry, they yield solutions with negative total masses.

## APPENDIX A: JUNCTION CONDITIONS

We shall use the Darmois-Israel formalism to determine the surface stresses at the junction boundary [25, 26]. Consider two spacetimes,  $M^+$  and  $M^-$ , with metrics  $g_{\mu\nu}^+(x_+^\lambda)$  and  $g_{\mu\nu}^-(x_-^\lambda)$ , defined respectively in the coordinate systems  $x_+^\lambda$  and  $x_-^\lambda$ . Assume that these spacetimes,  $M^+$  and  $M^-$ , have timelike boundaries  $\Sigma^+$  and  $\Sigma^-$ , respectively. The induced metrics on the boundaries are  $g_{ij}^+(\xi_+^k)$  and  $g_{ij}^-(\xi_-^k)$ , respectively, where  $\xi_\pm^k$  are the intrinsic coordinates on  $\Sigma_\pm$ . The Darmois-Israel formalism consists in pasting the spacetimes together, demanding that the boundaries are isometric, having the same coordinates,  $\xi_+^k = \xi_-^k = \xi^k$ . The identification  $\Sigma^+ = \Sigma^- \equiv \Sigma$  provides a single spacetime given by  $M = M^+ \cup M^-$ .

The parametric equation for  $\Sigma$  is given by  $f(x^\mu(\xi^i)) = 0$ , and the respective unit 4-normal to  $\Sigma$  is provided by  $n_\mu = \pm |g^{\alpha\beta} (\partial f / \partial x^\alpha) (\partial f / \partial x^\beta)|^{-1/2} (\partial f / \partial x^\mu)$ , with  $n^\mu n_\mu = +1$ . The intrinsic surface stress-energy tensor,  $S_{ij}$ , is given by the Lanczos equations [25] in the form  $S_j^i = -\frac{1}{8\pi} (\kappa_j^i - \delta_j^i \kappa_k^k)$ . For notational convenience, the discontinuity in the second fundamental form or extrinsic curvatures is given by  $\kappa_{ij} = K_{ij}^+ - K_{ij}^-$ . The second fundamental form is defined as

$$K_{ij}^\pm = \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \nabla_\alpha^\pm n_\beta = -n_\gamma \left( \frac{\partial^2 x^\gamma}{\partial \xi^i \partial \xi^j} + \Gamma_{\alpha\beta}^{\gamma\pm} \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \right). \quad (\text{A1})$$

The superscripts  $\pm$  correspond to the exterior and interior spacetimes, respectively.

Considerable simplifications occur due to plane symmetry, namely  $\kappa_j^i = \text{diag}(\kappa_\tau^\tau, \kappa_x^x, \kappa_x^x)$ . Thus, taking into account the Lanczos equations, the intrinsic surface stress-energy tensor may be written as  $S_j^i = \text{diag}(-\sigma, \mathcal{P}, \mathcal{P})$ , where the surface energy density,  $\sigma$ , and the surface pressure,  $\mathcal{P}$ , are given by

$$\sigma = -\frac{1}{4\pi} \kappa_x^x, \quad (\text{A2})$$

$$\mathcal{P} = \frac{1}{8\pi} (\kappa_\tau^\tau + \kappa_x^x). \quad (\text{A3})$$

This simplifies the determination of the surface stress-energy tensor to that of the calculation of the non-trivial components of the extrinsic curvature, or the second fundamental form.

In particular, we shall match the interior and exterior metrics, Eq. (2) and Eq. (14), respectively. The junction surface is situated at  $r = \bar{r} = a$ . In order for these line elements to be continuous across the junction surface,  $ds^2|_\Sigma = \bar{d}s^2|_\Sigma$ , we consider the following coordinate transformations:  $\bar{t} = t e^{\Phi(a)} / \sqrt{\bar{\alpha}^2 a^2 - M/\alpha a}$ ,  $d\bar{r}/dr|_{r=a} = \sqrt{\bar{\alpha}^2 a^2 - M/\alpha a} / \sqrt{\alpha^2 a^2 - m(a)/\alpha a}$ ,  $\bar{\alpha}\bar{x} = \alpha x$  and  $\bar{\alpha}\bar{y} = \alpha y$ .

Using Eq. (A1), the non-trivial components of the extrinsic curvature are given by

$$K_\tau^{\tau+} = \frac{1}{a} \frac{\bar{\alpha}^2 a^2 + \frac{\tilde{M}}{2\bar{\alpha}a}}{\sqrt{\bar{\alpha}^2 a^2 - \frac{\tilde{M}}{\bar{\alpha}a}}}, \quad (\text{A4})$$

$$K_\tau^{\tau-} = \Phi'(a) \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}}, \quad (\text{A5})$$

and

$$K_x^{x+} = K_y^{y+} = \frac{1}{a} \sqrt{\bar{\alpha}^2 a^2 - \frac{\tilde{M}}{\bar{\alpha}a}}, \quad (\text{A6})$$

$$K_x^{x-} = K_y^{y-} = \frac{1}{a} \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}}. \quad (\text{A7})$$

Thus, the Einstein field equations, Eqs. (A2)-(A3), with the extrinsic curvatures, then provide us with the following surface stresses

$$\sigma = -\frac{1}{4\pi a} \left( \sqrt{\bar{\alpha}^2 a^2 - \frac{\tilde{M}}{\bar{\alpha}a}} - \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}} \right), \quad (\text{A8})$$

$$\mathcal{P} = \frac{1}{8\pi a} \left[ \frac{2\bar{\alpha}^2 a^2 - \frac{\tilde{M}}{2\bar{\alpha}a}}{\sqrt{\bar{\alpha}^2 a^2 - \frac{\tilde{M}}{\bar{\alpha}a}}} - \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}} (1 + a\Phi'(a)) \right]. \quad (\text{A9})$$

One may also obtain an equation governing the behavior of the radial pressure in terms of the surface stresses at the junction boundary from the following identity

$$[T_{\hat{\mu}\hat{\nu}}^{\text{total}} n^{\hat{\mu}} n^{\hat{\nu}}] = \frac{1}{2}(K_j^{i+} + K_j^{i-}) S_j^i, \quad (\text{A10})$$

where  $T_{\hat{\mu}\hat{\nu}}^{\text{total}} = T_{\hat{\mu}\hat{\nu}} + g_{\hat{\mu}\hat{\nu}} 3\alpha^2/8\pi$  is the total stress-energy tensor, and the square brackets denotes the discontinuity across the thin shell, i.e.,  $[X] = X^+|_{\Sigma} - X^-|_{\Sigma}$ . Equation (A10) can also be deduced from the normal component of the conservation of the stress-energy tensor [19]. Taking into account the values of the extrinsic curvatures, and noting that the pressure acting on the shell is by definition the normal component of the stress-energy tensor,  $p_r = T_{\hat{\mu}\hat{\nu}} n^{\hat{\mu}} n^{\hat{\nu}}$ , we finally have the following pressure balance equation

$$\begin{aligned} \left(\bar{p}_r(a) + \frac{3\bar{\alpha}^2}{8\pi}\right) - \left(p_r(a) + \frac{3\alpha^2}{8\pi}\right) &= \frac{1}{a} \left( \sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}} + \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}} \right) \mathcal{P} \\ &\quad - \frac{1}{2} \left( \frac{\bar{\alpha}^2 a + \frac{M}{2\bar{\alpha}a^2}}{\sqrt{\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a}}} + \Phi'(a) \sqrt{\alpha^2 a^2 - \frac{m(a)}{\alpha a}} \right) \sigma, \end{aligned} \quad (\text{A11})$$

which relates the difference of the radial pressure across the shell in terms of a combination of the surface stresses and the geometrical quantities.  $\sigma$  and  $\mathcal{P}$  are given by Eqs. (A8)-(A9), respectively.

One may obtain Eq. (A11) in an alternative manner using directly the Einstein equations. The analysis is simplified considering two general solutions of Eq. (2), an interior and an exterior solution, written out in the coordinate systems,  $(t, r, x, y)$  and  $(\bar{t}, \bar{r}, \bar{x}, \bar{y})$ , respectively, and matched at  $\Sigma$ . Consider the radial component of the Einstein equations, Eq. (8), written out in both coordinate systems and note that in the exterior spacetime we have the following relationships:  $\bar{m} = M$  and  $\bar{\Phi}'(a) = (\bar{\alpha}^2 a - \frac{M}{2\bar{\alpha}a^2}) / (\bar{\alpha}^2 a^2 - \frac{M}{\bar{\alpha}a})$ . Taking into account the latter relations and the matching of the solutions given by Eq. (2) and Eq. (14), from the radial component of the Einstein equations, we deduce the Eq. (A11).

Working in the same coordinate systems, the continuity of the first fundamental form,  $ds^2|_{\Sigma} = \bar{ds}^2|_{\Sigma}$ , implies that the metric components are continuous, i.e.,  $g_{\mu\nu}^+ = g_{\mu\nu}^-$ . Thus, the equation governing the behavior of the radial pressure at  $\Sigma$ , Eq. (A11), reduces to the following form,  $\bar{p}_{0,r}(a) + 3\bar{\alpha}^2/8\pi = p_{0,r}(a) + 3\alpha^2/8\pi + 2\mathcal{P}_0 e^{\Phi(a)}/a$ , where we have defined  $e^{\Phi(a)} = \sqrt{\bar{\alpha}^2 a^2 - M/\bar{\alpha}a}$  and  $\mathcal{P}_0$  is given by Eq. (20).

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